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INTERFACIAL INSTABILITIES IN AN UNBONDED LAYERED SOLID

PAUL S. STEIF
Department of Mechanical Engineering, Carnegie-Mellon University,
Pittsburgh, PA 15213, U.S.A.

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Abstract—A bifurcation problem relevant to the process of roll-bonding is studied. The simultaneous rolling of several unbonded layers is modeled as a plane strain compression of the unbonded layers. To gain insight into the phenomenon of tiger banding, the strains at which there can exist bifurcation modes which involve an undulating interface between the layers are computed. Particular attention is focused upon the effect of permitting slip between the layers, in contrast to the assumption of velocity continuity that underlies previous analyses of bifurcation in layered solids. Velocity discontinuity implies that the bifurcation condition depends on the hydrostatic pressure; the consequences of this dependence are highlighted.

INTRODUCTION

Roll-bonding is a continuous process in which two or more sheets of different metals are fed simultaneously into a rolling mill. With intense straining and high pressure, the materials become bonded to one another. The exiting material is intended to be a multilayered clad metal with straight interfaces between the layers (see Fig. 1). It is necessary to adjust a wide range of processing variables, such as back and front tensions, percentage reduction, and lubrication, in order to achieve this desired product. A number of problems can arise when roll-bonding a new combination of metals for which the correct levels of reduction, lubrication, etc. are not yet known. One problem that can occur is a slight waviness of the interface, meaning that the relative thicknesses of the layers vary from point to point. This can, however, be so extreme as to cause the inner layers to break through the surface layer. The surface appearance, involving periodic bands of a second material, is sometimes referred to as tiger banding. Roll-bonding seems to have been explored very little by the academic community in the United States, and there appears to have been no attempts to understand tiger banding from a theoretical standpoint. In this paper, we present a theoretical model which may be capable of predicting this phenomenon.

The proposed model has its origins in previous work by the author on the emergence of internal, necking-type deformation modes in layered materials (Steif, 1986a,b, 1987a,b). [A brief partial review of related multi-material bifurcation problems is given in Steif (1986a).] Most relevant to the present work is a study of periodic undulating modes that can emerge during the rolling of *already bonded* clad metals (Steif, 1987a). That study was motivated by experimental observations of Semiatin and Piehler (1979), who subjected aluminum/stainless steel clad metals to various deformation states. In particular, they found that rolling these clad metals to rather high reductions can cause internal deformations which involve a severe waviness of the interface. In fact, for stainless steel symmetrically

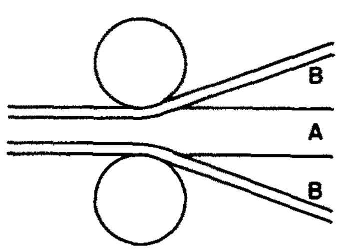

Fig. 1. Schematic of roll-bonding process.

clad on aluminum, they found that the waviness eventually leads to a periodic breaking of the aluminum through the steel (see Fig. 2). Steif (1987a) modeled the rolling as a plane strain compression; this loading *can* lead to a homogenous deformation of the clad sheet. It was found, however, that periodic deformation modes are also possible. Despite the fact that the plane strain compression grossly simplifies the deformation, which actually varies along the contact region in the roll gap (see, for example, Avitzur, 1968), the shape of the bifurcation modes, their wavelengths, and the reductions at which they emerge agreed roughly with the observations of Semiatin and Piehler (1979).

Here, it is proposed that tiger banding during *roll-bonding* can also be modeled as a bifurcation from a state of uniform plane strain compression. There is, however, a significant difference between the situations of roll-bonding and the rolling of *already bonded* clad metals. During a significant portion of the reduction between the rolls of a bonding mill, the individual layers are not yet bonded to one another. So while the layers may be fed in at the same rates, and the nominal reduction—the desired homogeneous deformation state—is the same in all layers, the layers are not *forced* to have identical displacements pointwise along their interface. This suggests that roll-bonding be modeled approximately as a plane strain compression of layers of different materials, with the layers permitted to slip relative to one another. However, the applied loading is such that all sheets *could* deform with the same uniform strain. This paper is devoted to assessing the effect of interfacial slippage on the emergence of internal necking-type bifurcations and the consequences for tiger banding during roll-bonding.

ANALYSIS

A schematic of the layered solid considered here is shown in Fig. 3. This solid is compressed by smooth rigid platens normal to the layering and is stretched parallel to the layering. The layers are not bonded to one another, though the loads are applied in such a way that they stretch the same amount. At a generic level of loading, the following question is posed: is there a deformation (involving interfacial waviness), consistent with the imposed conditions, *other* than the uniform straining? The ensuing analysis is devoted to determining the factors which affect the answer to this question.

The materials constituting the layers are assumed to be incompressible, rate-independent, incrementally orthotropic solids. As such, the material law can be expressed as (Hill and Hutchinson, 1975)

$$\overset{\nabla}{\sigma}_{11}-\overset{\nabla}{\sigma}_{22} = 2\mu^*(\varepsilon_{11}-\varepsilon_{22}) \tag{1a}$$

$$\overset{\nabla}{\sigma}_{12} = 2\mu\varepsilon_{12}, \tag{1b}$$

where $\overset{\nabla}{\sigma}_{\alpha\beta}$ (α and $\beta = 1, 2$) are the Jaumann derivatives of the Cauchy stress, $\varepsilon_{\alpha\beta}$ the Eulerian strain-rates, and μ and μ^* the incremental moduli.

Field equations governing incremental deformations from a finitely deformed state are well known. The stream function, from which the velocities are derived according to

$$v_1 = \frac{\partial\psi}{\partial x_2}, \quad v_2 = -\frac{\partial\psi}{\partial x_1}, \tag{2a,b}$$

must satisfy the equation

$$[\mu+\tfrac{1}{2}(\sigma_1-\sigma_2)]\psi_{,1111}+2(2\mu^*-\mu)\psi_{,2211}+[\mu-\tfrac{1}{2}(\sigma_1-\sigma_2)]\psi_{,2222} = 0 \tag{3}$$

where σ_1 and σ_2 are the principal Cauchy stresses (parallel to x_1 and x_2), and $(\ )_{,\alpha}$ denotes differentiation with respect to x_α.

Central to the present study are the velocities given by (2) and the nominal traction-rates given by

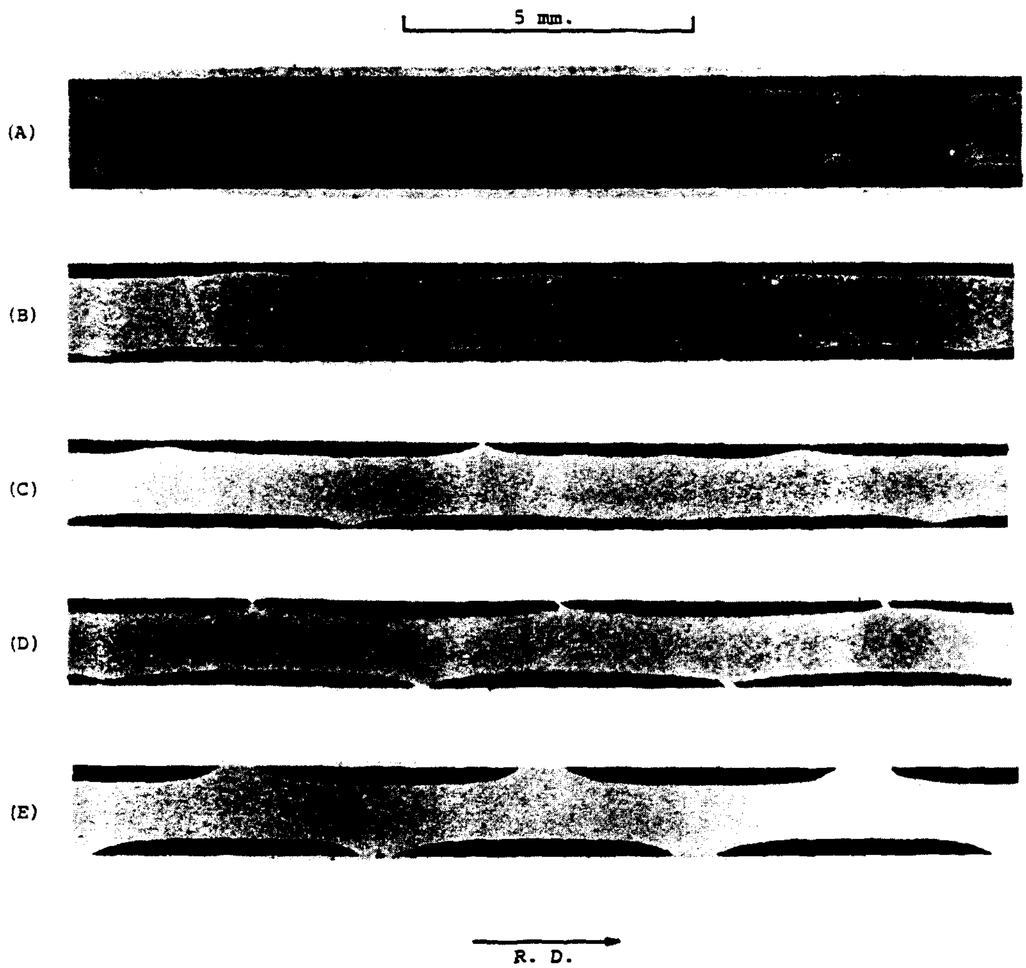

Fig. 2. Development of tiger bands during the rolling of an already bonded clad metal.

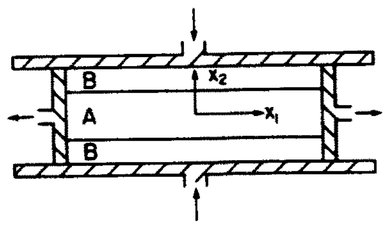

Fig. 3. Plane strain deformation of a layered solid.

$$\dot{n}_{21} = [\mu - \tfrac{1}{2}(\sigma_1 - \sigma_2)]\psi_{,22} - [\mu - \tfrac{1}{2}(\sigma_1 + \sigma_2)]\psi_{,11} \tag{4a}$$

$$-\dot{n}_{22,1} = [4\mu^* - \mu - \tfrac{1}{2}(\sigma_1 + \sigma_2)]\psi_{,211} + [\mu - \tfrac{1}{2}(\sigma_1 - \sigma_2)]\psi_{,222}. \tag{4b}$$

In previous work on bifurcations in layered materials, the layers were assumed to be perfectly bonded; continuity of traction-rate and velocity imply that

$$[\mu_A - \tfrac{1}{2}(\sigma_1^A - \sigma_2^A)]\psi^A_{,22} - [\mu_A - \tfrac{1}{2}(\sigma_1^A + \sigma_2^A)]\psi^A_{,11} = [\mu_B - \tfrac{1}{2}(\sigma_1^B - \sigma_2^B)]\psi^B_{,22} - [\mu_B - \tfrac{1}{2}(\sigma_1^B + \sigma_2^B)]\psi^B_{,11} \tag{5a}$$

$$[4\mu_A^* - \mu_A - \tfrac{1}{2}(\sigma_1^A + \sigma_2^A)]\psi^A_{,211} + [\mu_A - \tfrac{1}{2}(\sigma_1^A - \sigma_2^A)]\psi^A_{,222} = [4\mu_B^* - \mu_B - \tfrac{1}{2}(\sigma_1^B + \sigma_2^B)]\psi^B_{,211} + [\mu_B - \tfrac{1}{2}(\sigma_1^B - \sigma_2^B)]\psi^B_{,222} \tag{5b}$$

$$\psi^A_{,2} = \psi^B_{,2} \tag{5c}$$

$$\psi^A_{,1} = \psi^B_{,1} \tag{5d}$$

where subscripts (and superscripts) A and B denote quantities in the respective layers. Closer examination of eqns (5a) and (5b) reveals that they are dependent only on the combination $\sigma_1 - \sigma_2$ (in each of A and B). This can be demonstrated by rewriting $\sigma_1 + \sigma_2$ as $\sigma_1 - \sigma_2 + 2\sigma_2$, and then using (5c) and (5d) and the fact that σ_2 is identical in A and B. Since the principal strain directions do not change in the uniform pre-bifurcation states contemplated here, the combination $\sigma_1 - \sigma_2$ can generally be expressed simply in terms of the strain (for an incompressible material). Hence, bifurcation conditions for perfectly bonded clad materials depend only on the strains in the layers, and not on the particular combination of lateral compression and longitudinal tension.

Here, we seek to examine the effect of permitting the layers to slip relative to one another. This means that (5c) is removed; in its stead there must be an equation which relates the shear traction-rate to the other field quantities. As a preliminary investigation into the influence of slippage, we make perhaps the simplest assumption that slippage occurs unimpeded by friction. While this is certainly not true in rolling, this extreme assumption appears to give a bound on the effect we are seeking, because very high friction would be tantamount to perfect bonding. Thus, (5c) is replaced by the condition that the shear traction-rate vanishes at the interface; i.e.

$$[\mu_A - \tfrac{1}{2}(\sigma_1^A - \sigma_2^A)]\psi^A_{,22} - [\mu_A - \tfrac{1}{2}(\sigma_1^A + \sigma_2^A)]\psi^A_{,11} = 0. \tag{6}$$

A comment should be made regarding the assumption that normal velocities are continuous even though the layers are not bonded. Clearly, this is acceptable only if the normal traction is negative (compressive) all along the interface. However, the normal traction-rate associated with the bifurcation modes considered below varies sinusoidally with x_1 taking on both positive and negative values. Hence, it is necessary to restrict consideration to situations in which the pre-bifurcation normal stress σ_2 is compressive and the magnitude of the bifurcation mode (undetermined below) is sufficiently small. Although the limiting case of $\sigma_2 \to 0$ is considered below, this restriction should be borne in mind. The high pressures in a bonding mill make this restriction of little practical concern.

For simplicity we introduce the following parameters

$$R = \frac{\mu}{2\mu^*}, \quad S = \frac{\sigma_1}{4\mu^*}, \quad T = \frac{\sigma_2}{4\mu^*}, \quad \xi = \frac{\mu_B^*}{\mu_A^*}$$

where R, S and T will each be given subscripts A and B, so as to refer to the respective layers.

We now restrict the range of the parameters R and $S-T$ to be such that the incremental equations are elliptic: $S-T < \sqrt{2R-1}$. (This is generally satisfied by constitutive laws used for metals up to substantial strains, when the equations become hyperbolic.) Then, periodic bifurcation modes which are antisymmetric in the A layer, $-a/2 < x_2 < a/2$, and symmetric in each of the B layers, $a/2 < |x_2| < b/2$, (thereby preserving the boundary conditions at the smooth rigid platens) are of the form

$$\psi^A = \mathrm{Re}\left[c\cos\frac{2\pi}{\lambda}\alpha x_2\right]\sin\frac{2\pi x_1}{\lambda} \tag{7a}$$

$$\psi^B = \sin\frac{2\pi x_1}{\lambda}\begin{cases}\mathrm{Re}\left[d\sin\frac{2\pi}{\lambda}\beta\left(\frac{b}{2}-x_2\right)\right] & \frac{a}{2} < x_2 < \frac{b}{2}\\ \mathrm{Re}\left[d\sin\frac{2\pi}{\lambda}\beta\left(\frac{b}{2}+x_2\right)\right] & -\frac{b}{2} < x_2 < -\frac{a}{2}\end{cases} \tag{7b}$$

where α and β depend on the stresses and incremental moduli (see Steif, 1986a). c and d are undetermined complex amplitudes, λ is the wavelength and Re [ ] denotes the real part of the enclosed quantity. These forms for the stream function are substituted into (5a), (5b), (5d) and (6). Finally, one seeks values for the parameters R_A, S_A, T_A, R_B, S_B and ξ at which there exist nontrivial c and d satisfying the homogeneous equations. (Note that $T_B = T_A/\xi$.) A single bifurcation equation can be derived (as was done by Steif, 1986a,b) based on setting the determinant equal to zero. This equation is quite unwieldy and difficult to interpret. Instead, numerical results are presented for power-law hardening materials. These results, together with a closer examination of the continuity conditions at the interface, provide insight into the seminal features of bifurcation with slippage.

DISCUSSION AND NUMERICAL RESULTS

One issue of theoretical interest is the possibility of long-wavelength modes (long compared with the layer thicknesses). As found previously by this author, there are often not long wavelength bifurcation modes when the boundary conditions on surfaces parallel to the layering (on $x_2 = \pm b/2$ here) involve zero normal displacement. The explanation of this (Steif, 1986b; 1987a) is as follows: a long-wavelength mode involves a uniform deformation in each of the layers which, of course, is different from the pre-bifurcation uniform deformation. If the layers are all assumed to be incompressible and perfectly bonded (meaning that v_1 and v_2 are continuous), then there is no combination of nonzero uniform deformations of the layers that allows for there to be zero normal displacement at $x_2 = \pm b/2$.

If the velocity v_1 is no longer required to be continuous, then long-wavelength modes such as those contemplated here are possible. For example, the upper layer B can thin, the lower layer B can thicken, and the A layer can move rigidly upwards. Note that this mode preserves incompressibility, zero normal displacement at $x_2 \pm b/2$, and continuity of *normal* velocity at the interface. The possibility of such a mode and the conditions for its existence are readily seen by letting λ be large. The bifurcation equations then simplify to

$$\mathrm{Re}\,\{d^*[(1-\beta^2)(R_B-S_B)-(1+\beta^2)T_B]\} = 0 \tag{8a}$$

$$\mathrm{Re}\,\{d^*[\beta^2(R_B-S_B)-(R_B+S_B-2)+(\beta^2-1)T_B]\} = 0 \tag{8b}$$

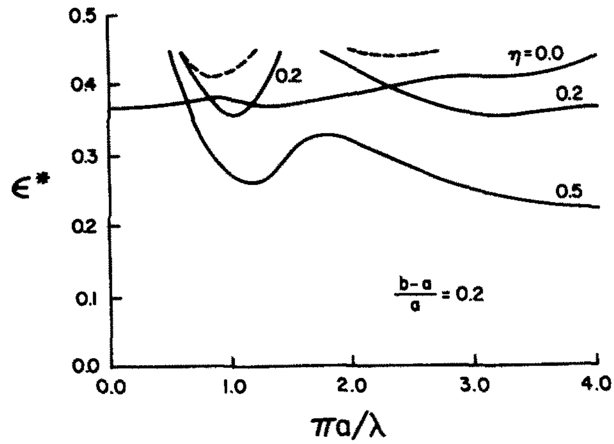

Fig. 4. Bifurcation strains for stainless steel clad aluminum with thin steel layers. ($N_A = 0.167$, $N_B = 0.389$, $k_B/k_A = 9.0$; - - - ~ perfectly bonded clad.)

where $d^* = d\beta$. Equations (8) correspond to a symmetric, long-wavelength mode in a block of material B subjected to biaxial stress with zero traction-*rates* prescribed on its surface ($x_2 = a/2$). (The symmetry axis is $x_2 = b/2$.) From the arguments offered above regarding the A layer moving rigidly in a long-wavelength mode, it ought to be expected that the bifurcation is indifferent to the properties of material A, and that layer A imposes zero traction-rate on layer B. Finite wavelength bifurcations of a rectangular block subjected to biaxial stresses will also be of interest below.

Specific numerical calculations were carried out assuming a power-law stress–strain relation which in uniaxial plane strain tension takes the form

$$\sigma = k\varepsilon^N. \tag{9}$$

In addition, it was assumed that the incremental modulus μ is given by the hyperelastic deformation theory of plasticity; hence,

$$\mu = \tfrac{1}{2}k\varepsilon^N \coth(2\varepsilon). \tag{10}$$

The incremental modulus μ, which governs the material response to shearing parallel to the principal axes, has been found to have a substantial influence on bifurcation calculations such as plastic buckling and necking (see Hutchinson, 1974). Choosing μ to be given by (10) reflects a consensus that using incremental moduli based on a smooth yield surface flow theory of plasticity results in too high a resistance to nonproportional strain increments.

With this material law, the parameters R, $S-T$ and ξ are given by

$$R = \frac{\varepsilon}{N}\coth(2\varepsilon), \quad S-T = \frac{\varepsilon}{N}, \quad \xi = \frac{k_B N_B \varepsilon^{(N_B-N_A)}}{k_A N_A}.$$

In addition, we introduce the parameter η which is a measure of the lateral pressure relative to the flow stress:

$$\eta = \frac{-T_B}{S_B - T_B}.$$

We have carried out computations for material parameters $N_A = 0.167$, $N_B = 0.389$, $k_B/k_A = 9.0$ which are intended to represent an aluminum symmetrically clad with stainless steel. Two cases were considered: (i) the aluminum is 80% of the total thickness, (ii) the stainless steel is 80% of the total thickness. The bifurcation strains ε^* for these cases are plotted in Figs 4 and 5, respectively. In Fig. 4, the curves representing the perfectly bonded

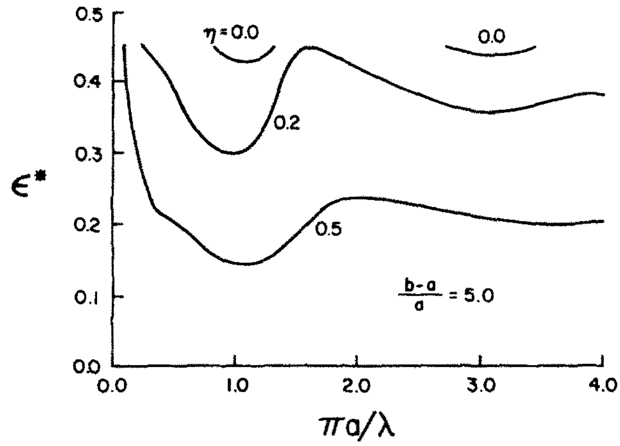

Fig. 5. Bifurcation strains for stainless steel clad aluminum with thick steel layers. ($N_A = 0.167$, $N_B = 0.389$, $k_B/k_A = 9.0$.)

case (the dashed line) and $\eta = 0$ should be considered first. It can be seen that allowing the layers to slip relative to one another significantly lowers the bifurcation strain relative to the perfectly bonded case. For case (ii), which is plotted in Fig. 5, there is not a bifurcation in the elliptic regime if perfect bonding between the layers is assumed. (Previous studies of layered materials have shown that an elliptic regime bifurcation is unlikely when the harder layer is thick relative to the more compliant layer.) However, with slippage allowed, a bifurcation is possible.

By focusing on curves in Figs 4 and 5 having $\eta \neq 0$, it can be seen that the dependence on the pressure is rather complicated. As η increases, longer wavelength modes occur at higher strains, and shorter wavelength modes occur at lower strains. In fact, the shorter wavelength modes can be made to occur at very low strains if η is sufficiently high. This rather complex pressure dependence is not associated solely with the multimaterial bifurcation problem. To see this, consider the bifurcation problem of a single rectangular block under biaxial stress [the generalization of Hill and Hutchinson's (1975) uniaxial problem]. For $N = 0.2$, the bifurcation strains corresponding to various values of η are plotted in Fig. 6; the complicated η-dependence clearly persists. In fact, concentrating only on infinitely short wavelength modes of the block (i.e. surface instabilities), one still finds the pressure dependence to be nontrivial (see Fig. 7).

Some insight into the effect of pressure on the bifurcation mode can be gained by considering the exclusion conditions derived by Hill and Hutchinson (1975). Let U be the velocity gradient potential from which the nominal traction-rates can be derived; that is,

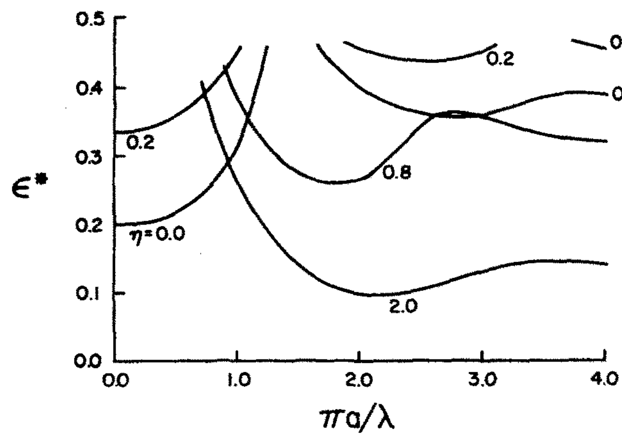

Fig. 6. Bifurcation strains for homogeneous block under biaxial loading ($N = 0.2$).

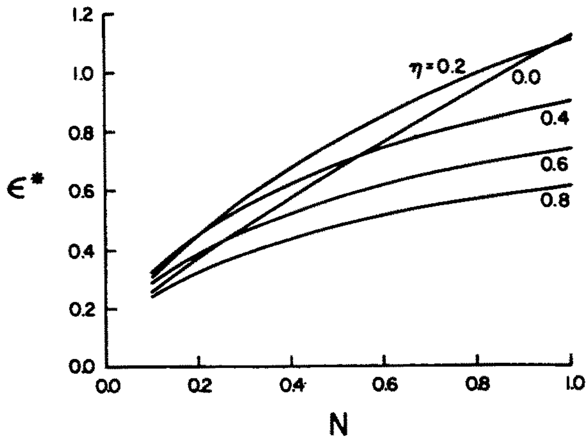

Fig. 7. Bifurcation strains for surface instabilities of a homogeneous block under biaxial loading.

$$\dot{n}_{ij} = \frac{\partial U}{\partial(v_{j,i})}.$$

Following Hill and Hutchinson (1975), and using the constitutive law (1), one can express U in the form

$$2U = 2[2\mu^* - \tfrac{1}{2}(\sigma_1+\sigma_2)]\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left[\sigma_1\frac{\partial v_2}{\partial x_1} - \sigma_2\frac{\partial v_1}{\partial x_2}\right]^2 \Big/ (\sigma_1+\sigma_2)$$
$$+\tfrac{1}{2}\left[2\mu - \frac{\sigma_1^2+\sigma_2^2}{\sigma_1+\sigma_2}\right]\left[\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right]^2.$$

If U is positive for all velocity gradients, then no bifurcation is possible, no matter what the boundary conditions. Hill (1967) has shown that a first-order bifurcation must occur just when this condition is violated when the boundary loading is dead. With mixed boundary conditions, a first order bifurcation may or may not occur. Nevertheless, in the circumstances here, the positive definiteness of U *at all points* does exclude bifurcations. Given that we are remaining in the elliptic regime, the relevant exclusion condition is

$$0 < \sigma_1+\sigma_2 < 4\mu^*. \tag{11}$$

If (11) holds at all points, then a bifurcation is impossible.

To see the consequences of this exclusion condition, consider bifurcations of the homogeneous block subjected to biaxial stresses which lead to a positive pre-bifurcation strain ε_{11}. The exclusion condition can then be recast in the convenient form

$$0 < (1-2\eta)(S-T) < 1 \tag{12}$$

where $S-T$ is a function only of the strain. (It equals ε/N for a power-law material.) It is also useful to point out that the maximum load condition (under biaxial stress) can be shown to be $S-T=1$. Hence, the introduction of lateral pressure T (<0), such that $0<\eta<0.5$, expands the range of strains for which a bifurcation *cannot* occur. Once $\eta > 0.5$, bifurcation is not automatically excluded for *any* positive strain (see Fig. 6).

Now, consider an application of the exclusion conditions to the layered material. In this case, the pre-bifurcation state is one that is piecewise constant. To exclude a bifurcation, the exclusion condition (11) must hold in *both* materials. Since, for the layered material, η was defined in terms of the quantities in B, one exclusion condition continues to be (12), where $S-T$ is S_B-T_B. Using continuity of σ_2 across the interface, one can write the second exclusion, $0 < S_A+T_A < 1$, in the form

$$0 < S_A - T_A - 2\xi\eta(S_B - T_B) < 1. \tag{13}$$

Equation (13) takes a more revealing form for power-law materials:

$$0 < \varepsilon\left[\frac{1}{N_A} - 2\xi\eta\frac{1}{N_B}\right] < 1. \tag{14}$$

Given appropriate values for N_A, N_B and k_B/k_A, in particular a large value of k_B/k_A, it is clearly possible to violate the exclusion condition for values of η less than 0.5. This is consistent with the relatively low bifurcation strains found for certain layered systems.

Finally, we note that a certain qualitative trend which was found to hold for perfectly bonded materials no longer holds when slippage is allowed. For example, say there is a thick compliant layer surrounded by two thin, hard layers (e.g. aluminum symmetric clad with stainless steel). Consider two bifurcation modes of the same wavelength: one is a mode of the layered system and one is a symmetric bifurcation mode of the thin layer (B) alone with zero incremental tractions on $x_2 = a/2$. When the layers are assumed to be perfectly bonded, the bifurcation strain at which the layered-system mode emerges was always found to be greater than the bifurcation strain of the mode involving B alone. That is, the presence of the other layer (A) always raised the bifurcation strain. However, it has been found here that once slippage is allowed, this is not necessarily the case: the bifurcation strains of B alone do not provide a lower bound to the bifurcation strains of the layered system.

CONCLUSIONS

The present paper represents an extension of previous work by the author aimed at understanding necking-type bifurcation modes in homogeneously deformed, layered media. With the particular goal of modeling the development of nonuniform layer thicknesses (tiger banding) during the process of roll-bonding, we have considered the consequences of permitting slippage between the layers (as can occur prior to bonding). It has been found that the relevant bifurcation modes can occur at strains that are significantly lower than the strains associated with similar modes in an identical medium with *perfectly bonded* layers. Thus, tiger banding ought to be of greater concern during roll-bonding than during the rolling of already bonded clad metals. Furthermore, it was found, in contrast to previous work on perfectly bonded layered media, that the bifurcation condition depends not only on the strain in the layers, but also on the lateral pressure (i.e. not just on the thickness reduction, but on the roll force as well). The dependence on the lateral pressure is complicated, however. Small amounts of lateral pressure raise the bifurcation strain, but larger pressures reduce the bifurcation strain. These observations suggest that the tendency for tiger banding to occur can be diminished by minimizing slippage between the layers (with adequate friction) and by eliminating unnecessarily high lateral pressures. Experiments to test the predictions of our modeling are being initiated, and the result will be reported in the near future.

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